

**STOCHASTIC MODELING OF BRANCHING PROCESS USED IN  
INSURANCE COMPANIES****<sup>1</sup>Md. Shohel Rana\*, <sup>2</sup>Rumana Rois**

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**ABSTRACT**

Choice of stochastic modeling attempts to model the decision process of an individual or segment in a particular context. Choice models are able to predict with great accuracy how individuals would react in a particular situation. Discrete choice models determine the probability that a decision maker will take a certain action based upon several attributes and choices. This study provides evidence on the evaluation of an individual's choice of alternative Insurance Companies. We have used some discrete choice stochastic models to assess the factors which influence on the evaluation of stochastic modeling in the context of Insurance Companies. Branching process model is used to construct a complete picture of Insurance companies' activities in terms of their earning pattern and subscriber's enrollment in their business. One of the most important factors in modeling of Insurance companies' activities is to forecast the future profit and number of client's involvement to know the long run performance of the company. This

information can be extract by stochastic modeling. Especially for Branching process we consider Rupali Life Insurance Company. The branching process is introduced and evaluated to get idea of the distribution of client's enrolment as well as it is also a stochastic population model based on explicit descriptions of individual life and reproduction .This model is also used in characterizing the distribution of the size of the population for different generations and the probability of extinction of the population.

**INDEX:** Introduction, Literature review, Illustration of Branching process (Data structure, Tree diagram for different generation), Probability of ultimate extinction.

## I. INTRODUCTION

A Branching process is a mathematical description of the growth of a population for which the individual produces off-springs according to stochastic laws. The Branching process was proposed by Galton and the probability extinction was first obtained by Watson by considering the probability generating function for the number of children in the  $n^{\text{th}}$  generation. This mathematical model was known as the Galton-Watson Branching Process and had been studied thoroughly in literature. (See, Jagers, 1975). In probability theory, a branching process is a Markov process that models a population in which each individual in generation  $n$  produces some random number of individuals in generation  $n + 1$ , according to a fixed probability distribution that does not vary from individual to individual. Branching processes are used to model reproduction; for example, the individuals might correspond to bacteria, each of which generates 0, 1, or 2 offspring with some probability in a single time unit. Branching processes can also be used to model other systems with similar dynamics, e.g., the spread of surnames in genealogy or the propagation of neutrons in a nuclear reactor.

A central question in the theory of Branching processes is the probability of ultimate extinction, where no individuals exist after some finite number of generations. It is not hard to show that, starting with one individual in generation zero; the expected size of generation equals  $\mu^n$  where  $\mu$  is the expected number of children of each individual. If  $\mu$  is less than 1, then the expected number of individuals goes rapidly to zero, which implies ultimate extinction with probability 1 by Markov's inequality. Alternatively, if  $\mu$  is greater than 1, then the probability of ultimate extinction is less than 1 (but not necessarily zero; consider a process where each individual either dies without issue or has 100 children with equal probability). If  $\mu$  is equal to 1, then ultimate extinction occurs with probability 1 unless each individual always has exactly one child. In theoretical ecology, the parameter  $\mu$  of a branching process is called the basic reproductive rate.

## 2. Literature Review

### 2.1 Branching model

Consider an organism, cell, particle or an individual, whose life time has length one unit. At the end of its life time, it produces a random number  $\xi$  of off-spring where

$$P[\xi = k] = a_k, (k = 0, 1, \dots), (\sum a_k = 1) \quad (2.1)$$

Each one of the off-spring at the end of its life time produces off-springs the number of which has the same distribution.

Let be the number of particles at the end of the generation. We shall take since there is only one individual initially. The number of particles in the generation is the off-spring of those of the generation. If and denote the off-spring of the individuals.

$$P[X_{n+1} = j | X_{n1} = i_1, X_{n2} = i_2, \dots, X_n = i], \quad n_1 < n_2 < \dots < n, \\ = P[X_{n+1} = j | X_n = i] = p_{ij}^{(n)} \quad (2.2)$$

This equation depends on  $i$  and the off-spring distribution is Markov chain with state space  $(0, 1, 2, \dots)$ .

The model as given above is called Galton-Watson branching process. (See, Baht, 2000).

## 2.2 Assumptions of Branching process

Even though some of the assumptions such as independent identical off-spring distributions, on-overlapping of generations, are stringent, because of the simplicity of the model Branching process has found applications in many areas. (See, Baht, 2000).

- i. The number of individuals initially present, denoted by is called the size of the 0<sup>th</sup> generation.
- ii. All offspring of the 0<sup>th</sup> generation constitute the first generation and their number is denoted by.
- iii. In general denote the size of the  $n^{\text{th}}$  generation and follows that  $\{X_n\}$  is a Markov chain having as its state space that the set of non negative integers.
- iv. To find the characteristics of the distribution of the population for different generation.
- v. To find the probability of extinction of the population.

### 2.3 Mean and Variance of the branching process

For  $X_0 = 1$ ,  $m = E(X_1) = E(k) = A'$  is the off-spring mean. Let  $\sigma^2 = \text{Var}(X_1) = \text{Var}(k)$  be its variance. Then

$$E(X_{n+1}) = \phi'_{n+1}(1) = \phi'_n(A(s))A'|_{s=1} = m^{n+1}, n=1, 2, \dots \quad (2.3)$$

If  $X_0 = i$ , then  $E(X_{n+1}) = i m^{n+1}$ .

Variance of  $X_n$  is

$V(X_n) = \phi''_n(1) + \phi'_n(1) - [\phi'_n(1)]^2$ , thus

$$V(X_n) = \sigma^2 m^{n-1} \left( \frac{m^n - 1}{m - 1} \right), \text{ if } m \neq 1$$

$$= n\sigma^2 \text{ if } m=1 \quad (2.4)$$

If  $X_0 = i$ , then  $E(X_n) = i m^n$ .

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$$= in\sigma^2 \text{ If } m=1. \quad (2.5)$$

### 2.4 Determining the Probability of Extinction of the Population

Let  $\pi_0$  denote the probability of die out (under the assumption that  $X_0 = 1$ ). More formally,

$$\pi_0 = \lim_{n \rightarrow \infty} P\{X_n = 0 | X_0 = 1\}$$

The problem of determining the value of  $\pi_0$  was first raised in connection with the extinction of family surnames by Galton in 1889.

We first note that  $\pi_0 = 1$  if  $\mu < 1$ . this follows since

$$\mu^n = E(X_n) = \phi'_{n+1}(1) = \phi'_n(A(s))A'|_{s=1}, \quad n = 1, 2, \dots$$

Since  $\mu^n \rightarrow 0$  when  $\mu < 1$ ,

Then the Probability of Extinction of the Population is given by

$$P\{X_n = 0\} \rightarrow 1$$

In fact it can be shown that  $\pi_0 = 1$ , when  $\mu = 1$ . When  $\mu > 1$ , it turns out that  $\pi_0 < 1$ , and an equation determining  $\pi_0$  may be derived by conditioning on the number of offspring of the initial individual as follows

$$\pi_0 = \sum_{k=0}^{\infty} P\{\text{Population dies out} \mid X_1 = k\} P_k$$

No, given that  $X_1 = k$ , the population will eventually die out if and only if each of the  $k$  families started by the members of the first generation eventually dies out.

Since each family is assumed to act independently and since the probability that any particular family dies out is just  $\pi_0$ , this yields

$$P\{\text{Population dies out} \mid X_1 = k\} = \pi_0^k$$

And thus  $\pi_0$  satisfies

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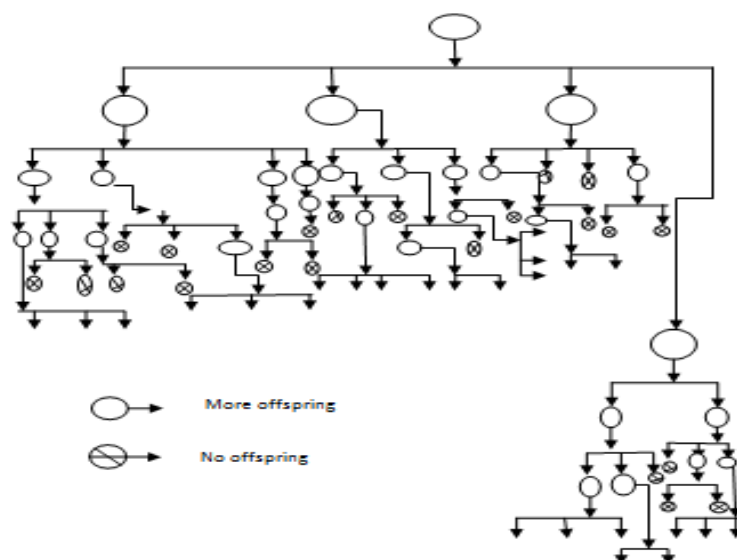
$$\pi_0 = \sum_{k=0}^{\infty} \pi_0^k P_k \quad (2.6)$$

In fact when  $\mu > 1$ , it can be shown that  $\pi_0$  is smallest positive number.

### 3. RESULT AND ANALYSIS

#### 3.1 Data structure

This data structure is taken from Rupali Life Insurance Company (RLIC). Which shows that how many individuals are involved in this company under one individual to another individual as a branching way. In the following figure empty circle indicates that individuals are further included and on the other hand cross circle indicates that individuals are not further included.



**Figure 3.1: Data structure of involved individuals in Rupali Life Insurance Company.**

**Table 3.1: Total number of clients involved under each immediate senior client**

S.N. of clients	Subscribed no. of clients	S.N. of clients	Subscribed no. of clients	S.N. of clients	Subscribed no. of clients	S.N. of clients	Subscribed no. of clients	S.N. of clients	Subscribed no. of clients
1	4	11	0	21	0	31	2	41	0
2	4	12	0	22	2	32	0	42	0
3	3	13	0	23	3	33	0	43	0
4	4	14	1	24	3	34	4	44	3
5	2	15	1	25	3	35	0	45	2
6	3	16	3	26	2	36	2	46	0
7	0	17	2	27	2	37	0	47	0
8	3	18	2	28	0	38	3	48	2
9	0	19	2	29	0	39	0	49	3
10	0	20	0	30	2	40	2	50	2

**Table 3.2: Distribution of offspring of the surveyed individual involved in Rupali Life Insurance Company.**

No. of offspring (X)	Frequency	Probability (P <sub>i</sub> )
0	20	0.4
1	2	0.04
2	14	0.28
3	10	0.2
4	4	0.08
<b>Total</b>	<b>50</b>	<b>1</b>

Mean of the down line clients (offspring) is calculated below:

$$\mu = E(X_1) \quad (3.1)$$

$$\begin{aligned} &= \sum_{k=0}^4 k p_k \\ &= 0 \times p_0 + 1 \times p_1 + 2 \times p_2 + 3 \times p_3 + 4 \times p_4 \\ &= 0 \times 0.4 + 1 \times .04 + 2 \times .28 + 3 \times .20 + 4 \times .08 = 1.52 \end{aligned}$$

Hence, the variance is;

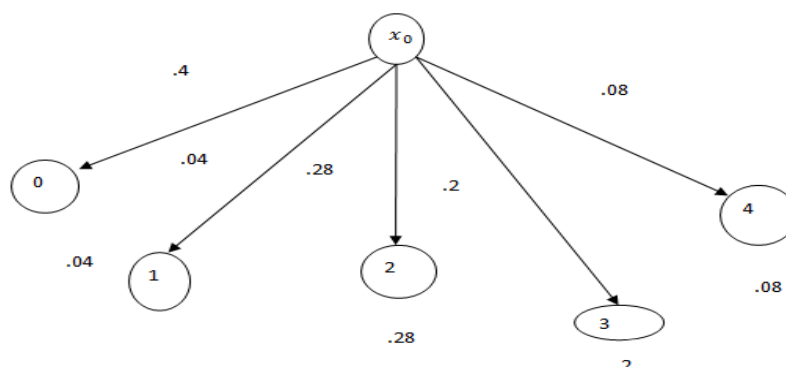
$$\sigma^2 = \text{var}(X_1) = E(X_1^2) - \{E(X_1)\}^2 \quad (3.2)$$

$$\begin{aligned} &= (0^2 \times p_0 + 1^2 \times p_1 + 2^2 \times p_2 + 3^2 \times p_3 + 4^2 \times p_4) - (\mu)^2 \\ &= (0^2 \times .4 + 1^2 \times .04 + 2^2 \times .28 + 3^2 \times .20 + 4^2 \times .08) - (1.52)^2 \\ &= 1.93 \end{aligned}$$

Since our calculated mean is  $\mu=1.52$  which is greater than 1, hence we can say that our obtaining Branching process is supercritical.

### 3.3 Different Generations Structure of down line Clients

At the first step, we can observe the 1<sup>st</sup> generation client's structure by tree diagram in branching process.



**Figure 3.2: Tree diagram of the first generation offspring distribution with their corresponding probabilities.**

Figure 3.2 represents that there is maximum chance (40%) to enroll zero clients followed by 28% with two clients, 20% with three clients in the first generation and only 4% associate with one client.

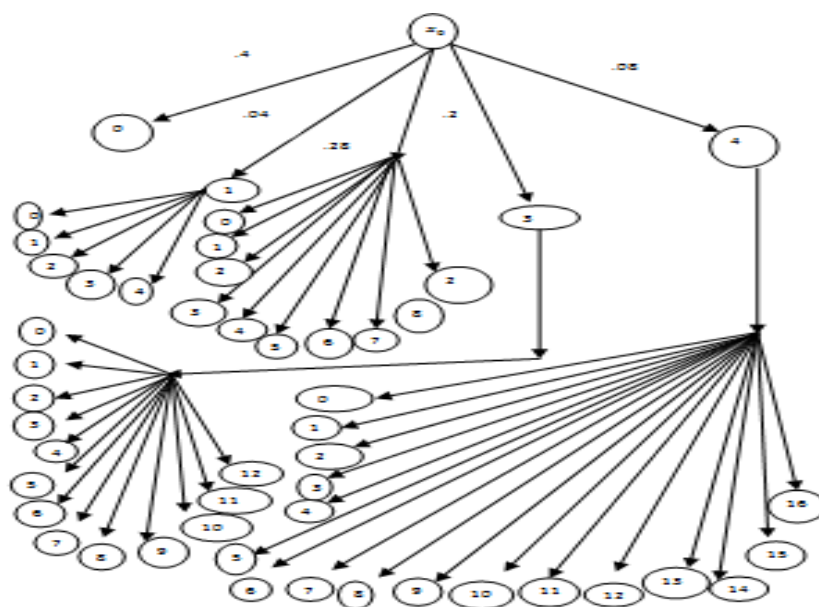


Figure 3.3: Tree diagram for the second generation of the offspring distribution with their corresponding probabilities.

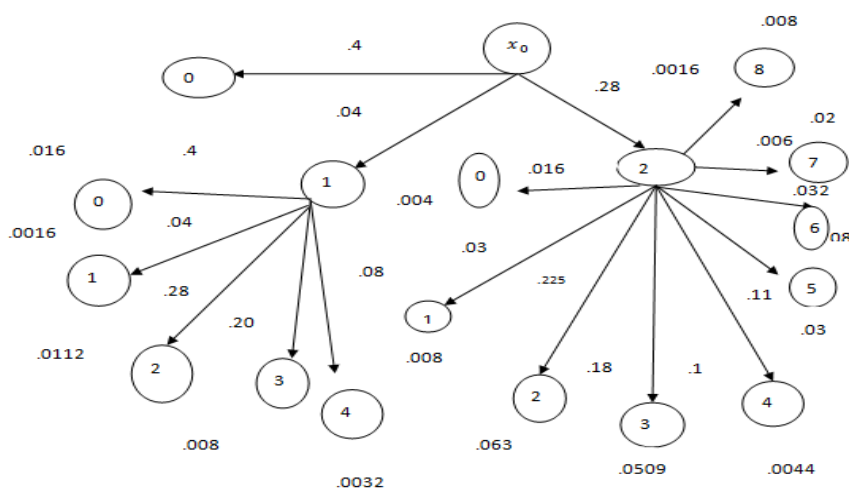


Figure 3.4: Tree diagram for the second generation of the offspring distribution for one and two individuals with their corresponding probabilities.

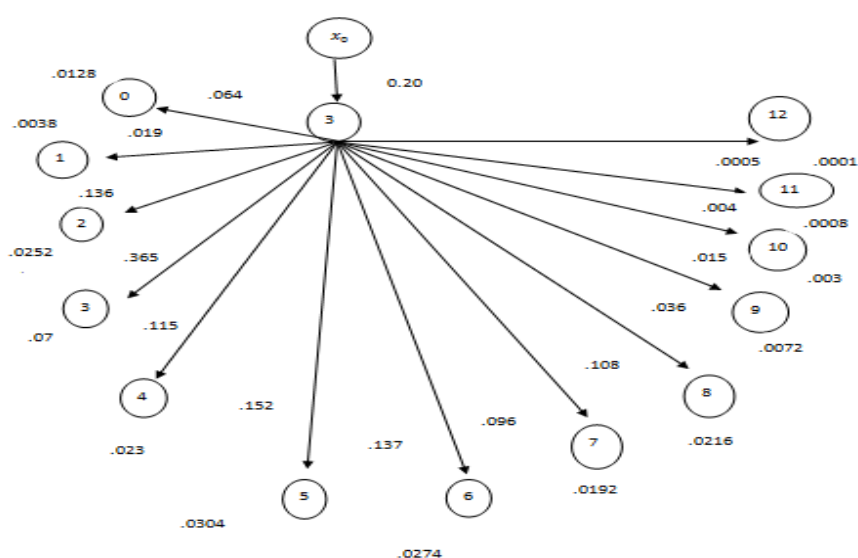
Table 3.3: Probability of offspring distribution for the second generation for one individual:

Probability( $p_i$ )	Probability values of number of $i^{th}$ offspring
$p_0$	.016
$p_1$	0.0016
$p_2$	0.112
$p_3$	0.008
$p_4$	0.0032

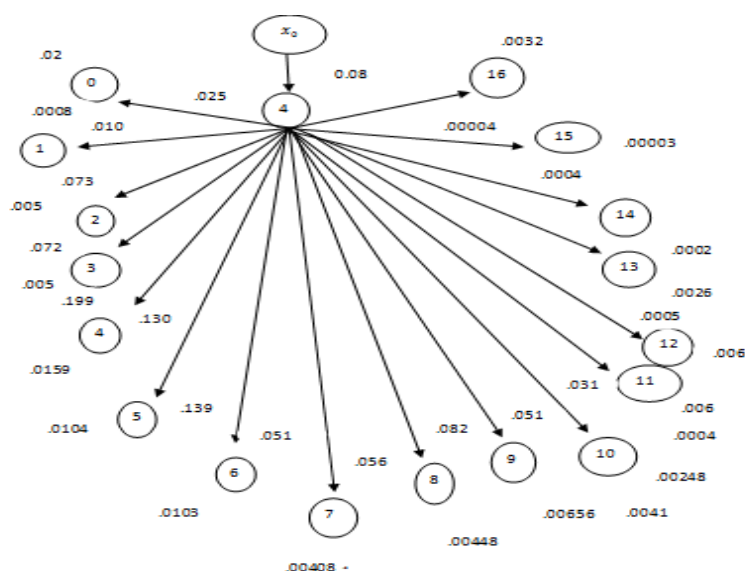


**Table 3.4:** Probability distribution of offspring for the second generation of two individuals:

Probability ( $p_i$ )	Probability values of number of $i^{th}$ offspring
$p_0$	0.0044
$p_1$	0.0089
$p_2$	0.063
$p_3$	.0509
$p_4$	0.0044
$p_5$	0.0330
$p_6$	0.0224
$p_7$	0.0089
$p_8$	0.0016

**Figure 3.5:** Tree diagram for the second generation of the offspring distribution for three individuals with their corresponding probabilities.**Table 3.5:** Probability of offspring distribution for the second generation of three individuals

Probability ( $p_i$ )	Probability values of number of $i^{th}$ offspring
$p_0$	0.0128
$p_1$	0.0038
$p_2$	0.0252
$p_3$	0.073
$p_4$	0.023
$p_5$	0.0304
$p_6$	0.0274
$p_7$	0.0192
$p_8$	0.0216
$p_9$	.0072
$p_{10}$	.003
$p_{11}$	.0008
$p_{12}$	.0001



**Figure 3.6:** Tree diagram for the second generation of the offspring distribution for four individuals with their corresponding probabilities.

**Table 3.6:** Probability of offspring distribution for the second generation of four individuals

Probability ( $p_i$ )	Probability values of number of $i^{th}$ offspring	Probability ( $p_i$ )	Probability values of number of $i^{th}$ offspring
$p_0$	.02	$p_8$	.00448
$p_1$	.0008	$p_9$	.00656
$p_2$	.00589	$p_{10}$	.0041
$p_3$	.00576	$p_{11}$	.00248
$p_4$	.01592	$p_{12}$	.00048
$p_5$	.0104	$p_{13}$	.0005
$p_6$	.0103	$p_{14}$	.0002
$p_7$	.00408	$p_{15}$	.000032
		$p_{16}$	.0032

We found that the probability of large offspring size is near to 0 for future generations. To establish the similar findings we can calculate the mean and variance of different generations.

We know that,

$$E(X_n) = \mu^n \quad (3.3)$$

$$\text{Variance}(X_n) = \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1} \sigma^2 \quad \text{if } \mu \neq 1 \quad (3.4)$$

And probability of ultimate extinction

$$u_n = \sum_{k=0}^{\infty} p_k u_{n-1}^k$$

$$u_n = p_0 u_{n-1}^0 + p_1 u_{n-1}^1 + p_2 u_{n-1}^2 + p_3 u_{n-1}^3 + p_4 u_{n-1}^4$$

$$u_n = .40 u_{n-1}^0 + .04 u_{n-1}^1 + .28 u_{n-1}^2 + .20 u_{n-1}^3 + .08 u_{n-1}^4$$

$$u_n = .40 + .04 u_{n-1}^1 + .28 u_{n-1}^2 + .20 u_{n-1}^3 + .08 u_{n-1}^4 \quad (3.5)$$

Now if  $n=1$  then  $u_0 = 0$  so substituting the different values of  $n=1, 2, \dots$  we get,

The probability of ultimate extinction for 20<sup>th</sup> generation using equation (3.5) as follows:

**Table 3.7: Mean variance and probability of ultimate extinction of Down Line Clients for Different Generations until 20<sup>th</sup> generation using equation (3.3), (3.4) and (3.5) respectively.**

Generat ion  Number	Mean	Variance	Probabil ity of ultimate extinctio n	Gen erati on  Num ber	Mean	Variance	Probabilit y of ultimate extinctio n
1	1.52	1.93	0.40	11	100.064	24205.124	0.5433
2	2.31	7.392	0.48	12	152.097	56116.644	0.5434
3	3.51	21.539	0.51	13	231.188	129945.445	0.54348
4	5.34	56.541	0.525	14	351.406	300672.150	0.5435
5	8.11	140.936	0.535	15	534.138	695351.152	0.54353
6	12.33	341.278	0.541	16	811.890	1607570.19	0.54355
7	18.75	812.292	0.542	17	1234.07	3715697.11	0.54356
8	28.49	1912.899	0.5426	18	1875.79	8587128.37	0.54357
9	43.31	4474.555	0.5430	19	2851.20	19843321.6	0.54357
10	65.83	10421.60	0.5432	20	4333.82	45851513.2	0.54358

So the probability of extinction is .54. Since the probability of extinction is not very high so we think that the insurance company has not a great chance to abolish soon. So it will be better for customers to make any further policy with this company.

#### 4. CONCLUSION

Under this study we have observed from the tree diagrams or down line structures for different generations using the branching process that the number of client's enrolment will be increased gradually in Rupali Life Insurance Company from one generation to the further more generation for a long period until  $n^{\text{th}}$  generation.

Again we have examined from the mean variance and probability of ultimate extinction table that the down line client's mean is 1.52 and variance is 1.93 and probability of ultimate extinction is 0.40 of Rupali Life Insurance Company for the first generation. Whereas the down line client's mean is 2.3104 and variance is 7.392 and probability of ultimate extinction is 0.48 for the second generation and so on that means mean, variance and as well as the probability of ultimate extinction is increasing from the first generation to the further more generations until 20<sup>th</sup> generation and thus the probability of ultimate extinction at or prior to 20<sup>th</sup> generation is near about  $u_{20} = .543$ ,

Mean and  $var(x_{20}) = 45851513.24$  and it will be gradually increased for a long period until  $n^{\text{th}}$  generation. Again according to Galton Watson process (Branching process) we know that if  $\mu > 1$ , branching process will be super critical. Since the resulting mean of the branching process used in Rupali Life Insurance Company is  $\mu = 1.52 > 1$  for the 1<sup>st</sup> generation and it will be same for further more generations hence we can say that the branching process is super critical. Further we have also found from the solution of quadratic equation that the probability of extinction is 0.54. Since the probability of extinction is not very high so we can draw an inference that the insurance company has not a greater chance to abolish soon. So it will be better for clients to make any further policy with this company.

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